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ASYMPTOTIC MODELLING OF RIGID BOUNDARIES AND CONNECTIONS IN THE RAYLEIGH–RITZ METHOD

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1. INTRODUCTION

The Rayleigh–Ritz method has proven to be a popular and powerful method for determining the natural frequencies and mode shapes of free vibration of continuous systems. When formulated using the potential and kinetic energies of the system, the method requires only that the chosen displacement functions (admissible functions) form a complete series and be able to satisfy the geometric boundary conditions of the system in order that the solution should converge toward the exact natural frequencies. The convergence is almost exclusively from above, although, as pointed out by Bhat [1], the Rayleigh–Ritz optimisation process renders the natural frequencies stationary but not necessarily minima and hence upper bounds. However, the acknowledged effective upper bound characteristic that does exist is a very important and useful feature of the method.

It is sometimes difficult or inconvenient to choose admissible functions that satisfy the geometric boundary conditions of rigidly supported or rigidly connected systems. Following the work of Courant [2], some researchers have treated such systems by using fully free or partially restrained admissible functions and satisfying the required rigid support and/or rigid connection conditions through the introduction of what are sometimes called "artificial springs" (although a better description may be "imaginary springs"). The stiffness of these springs (translational and/or rotational) is permitted to become very high, thus approximating rigid connections or supports. (This can be viewed as a physical interpretation of the penalty function method [3].) Examples of this approach for single plates are found in the works by Filipich *et al*. [4], Warburton and Edney [5], Gorman [6] and by Kim *et al*. [7], in which simply supported and clamped rectangular plates were treated as limiting cases of spring supported plates. The usefulness of the approach for the treatment of connected systems is demonstrated in the works by Yuan and Dickinson on straight and curved beam systems [8], rectangular plate systems [9], circular and annular plate/cylindrical shell systems [10] and circular, annular and sectorial plate systems [11, 12]. The approach was also used by Cheng and Nicolas [13] for the study of circular plate/cylindrical shell

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systems, Lee and Ng [14, 15] for cracked and stepped beams and by Young and Dickinson for connected, general sectorial plates [16] and shallow shells [17, 18].

As mentioned, the correct application of a Rayleigh–Ritz solution for continuous systems almost always yields upper bounds on the natural frequencies. With the introduction of imaginary springs to approximate rigid supports and/or connections, this useful characteristic is maintained for the approximate (slightly more flexible) model but not necessarily for the true, rigidly supported and/or connected system. In the cases cited above, the upper bound characteristic for the original systems was essentially preserved as it was possible to use sufficiently high values of stiffness for the springs, without encountering numerical difficulties in the computations, such that the achieved degree of convergence from below (as the spring stiffness was increased) was considerably higher than that achieved in the convergence from above (as the number of terms in the series was increased). In general, however, the relative degree of convergence achievable by increasing the spring stiffnesses and the number of terms in the displacement series will be dependent upon the nature of the system, the nature of the chosen displacement functions and the mode in question. For example, if an analyst were fortunate enough to choose a set of displacement functions that were exact for the original system, then no matter what finite value of stiffness were to be used for the imaginary springs, the solution would always be below that for the true system.

It would clearly be of benefit to modify the approach in order that the effective upper bound characteristic is maintained. This can be done simply by the introduction of imaginary springs of negative instead of positive stiffness. A physical interpretation, though not essential, is interesting and useful to consider. The introduction of a spring of negative stiffness can be construed as equivalent to the introduction of a point mass of value "stiffness/ω^2" (where ω is the circular frequency), since the "force of restraint" in one case is "$-$stiffness $\times$ displacement" and in the other "$-$mass $\times$ ω^2 $\times$ displacement". Here the displacement must be interpreted as the relative displacement experienced by the spring, whether it is approximating a rigid boundary or a connection between two components.

In order to illustrate the behaviour of the solution for systems asymptotically modelled in this manner, a clamped–simply supported beam and a simply supported stepped beam are considered. Comparisons of results obtained by using positive and negative spring stiffnesses with those from exact and/or Lagrangian Multiplier solutions show the error introduced by the asymptotic modelling. The emphasis here is on the methods of analysis rather than on the systems considered. Therefore, in order to keep the article short, the approaches used are described but the fairly lengthy equations that result are not given.

2. ILLUSTRATIVE EXAMPLES

2.1. *Natural frequencies of a clamped-simply supported beam*

In order to illustrate the behaviour of a system with a support approximated by a spring of positive and negative stiffness, an Euler–Bernouilli beam of length L, flexural rigidity EI, cross sectional area A and density ρ, clamped at one end and laterally supported at the other by a spring of stiffness k with no rotational constraint (ultimately to approximate a simple support), is considered. The exact solution for the problem is readily obtainable from the beam differential equation with substitution of the appropriate boundary conditions. Table 1 shows the non-dimensional frequency parameter $\lambda = (\rho A L^4 \omega^2 / EI)^{1/4}$ for the first few modes of vibration, as computed using this exact solution, with increasing values of positive and negative spring stiffness parameter $K = kL^3/EI$, together with the exact solutions for a clamped-simply supported beam [19]. It may be seen that as the absolute value of the spring stiffness parameter is increased, the values of the frequency parameters converge toward those for a clamped-simply supported beam, approaching from below for the positive spring stiffness and from above for the negative stiffness, as would be expected. The bounds on the solution obtained with any particular value of the modulus of K are immediately evident.

TABLE 1

Exact natural frequency parameters for clamped-partially restrained beam

	Frequency parameter		
K	λ_1	λ_2	λ_3
−1E + 09		3·92660	7·06858
−100000000		3·92660	7·06858
−10000000		3·92661	7·06860
−1000000		3·92663	7·06876
−100000		3·92689	7·07035
−10000		3·92946	7·08610
−1000		3·95502	7·22632
−100		4·17660	7·67558
−10		4·60144	7·83439
−2·99	0·45383	4·66551	7·84861
−2·95	0·67857	4·66589	7·84869
−2·9	0·80686	4·66636	7·84880
−2·8	0·95929	4·66731	7·84900
−2·4	1·26129	4·67110	7·84982
−2	1·43171	4·67490	7·85064
−1	1·69848	4·68446	7·85270
0	1·87510	4·69409	7·85476
1	2·01000	4·70379	7·85682
10	2·63892	4·79377	7·87565
100	3·64054	5·61600	8·08409
1000	3·89780	6·87629	9·55253
10000	3·92374	7·05070	10·15498
100000	3·92632	7·06681	10·20483
1000000	3·92657	7·06841	10·20964
10000000	3·92660	7·06856	10·21012
100000000	3·92660	7·06858	10·21017
1000000000	3·92660	7·06858	10·21018
∞ [19]	3·92660	7·06858	10·21018

It is interesting to note that as K tends towards -3, where the absolute value of the stiffness becomes that of a tip loaded cantilever, the fundamental frequency tends towards zero; this represents a flutter type of instability. (If the added mass interpretation is put on the use of a spring of negative stiffness, then this corresponds to the mass tending towards infinity and the circular frequency tending towards zero, the non-dimensional product of the mass and ω^2, tending towards 3.) When using the imaginary spring of negative stiffness to approximate a rigid support or connection, the risk of the introduction of zero frequency modes is negligible as the absolute value of the stiffness would always be chosen to be orders of magnitude larger than that that could lead to such a case.

A Rayleigh–Ritz solution for the clamped-spring supported problem may be formulated as follows. The lateral displacement of the beam may be expressed by the series of simple polynomials

$$f(x) = \sum_{j=1}^{n} a_j (x/L)^{j+1}, \tag{1}$$

where n is the number of terms used. These functions satisfy the zero slope and displacement condition at the clamped end ($x = 0$) but permit slope and deflection to exist at the spring supported end ($x = L$). When substituted into the Rayleigh–Ritz minimisation equation,

$$\partial V/\partial a_i - \omega^2 \partial \psi/\partial a_i = 0, \tag{2}$$

where the total potential energy V due to vibration and the kinetic energy function ψ are given by

$$V = \int_0^L \frac{EI}{2} (f''(x))^2 \, \mathrm{d}x + \tfrac{1}{2} k \cdot f(L)^2, \qquad \Psi = \int_0^L \frac{m}{2} (f(x))^2 \, \mathrm{d}x. \tag{2a, b}$$

A matrix eigenvalue problem of the standard form results which can be solved using any one of a number of standard algorithms. (For this work, the eliminant was formed and the roots determined using a Newton–Raphson approach.) As mentioned before, the clamped-simply supported case is approached by letting the modulus of k become very large.

The Lagrangian multiplier method is useful for comparison purposes, since it identically satisfies the rigid support condition and can be constructed using the same shape functions. Here, an additional constraint equation is required,

$$f(L) = \Sigma\, a_j (L/L)^j = \Sigma\, a_j = 0, \tag{3}$$

and the deletion of the k term in equation (2a). The modified Rayleigh–Ritz equation then becomes

$$\partial V/\partial a_i - \omega^2 \partial \psi/\partial a_i + \partial (a_{n+1} \cdot f(L))/\partial a_i = 0, \tag{4}$$

in which the Lagrangian multiplier is denoted by a_{n+1}. The matrix equation that results is not of the standard eigenvalue form but, again, the natural frequencies may be determined by searching for the roots of the eliminant.

TABLE 2

Comparison of natural frequency parameters for the clamped-restrained beam

		|K|				
n	Sign of K	10^5	10^6	10^7	10^8	Lagrangian
		First Mode: $\lambda_1 = 3{\cdot}92660$*				
5	+	3·92635	3·92661	3·92664	3·92664	3·92664
	−	3·92643	3·92667	3·92664	3·92664	
6	+	3·92632	3·92658	3·92660	3·92661	3·92661
	−	3·92689	3·92664	3·92661	3·92661	
7	+	3·92632	3·92657	3·92660	3·92660	3·92660
	−	3·92689	3·92663	3·92661	3·92660	
8	+	3·92632	3·92657	3·92660	3·92660	3·92660
	−	3·92689	3·92663	3·92661	3·92660	
		Second Mode: $\lambda_2 = 7{\cdot}0686$*				
5	+	7·0966	7·0983	7·0985	7·0985	7·0985
	−	7·1004	7·0987	7·0986	7·0986	
6	+	7·0687	7·0694	7·0696	7·0696	7·0696
	−	7·0713	7·0697	7·0696	7·0696	
7	+	7·0671	7·0687	7·0688	7·0688	7·0689
	−	7·0706	7·0690	7·0689	7·0689	
8	+	7·0668	7·0684	7·0686	7·0686	7·0686
	−	7·0704	7·0688	7·0686	7·0686	
		Third Mode: $\lambda_3 = 10{\cdot}210$*				
5	+	10·422	10·428	10·428	10·428	10·428
	−	10·434	10·429	10·428	10·428	
6	+	10·344	10·350	10·351	10·351	10·351
	−	10·357	10·351	10·351	10·351	
7	+	10·213	10·217	10·218	10·218	10·218
	−	10·223	10·218	10·218	10·218	
8	+	10·208	10·213	10·213	10·213	10·213
	−	10·219	10·214	10·213	10·213	

* "Exact value" from the solution of the characteristic equation [19].

Table 2 shows a convergence study for the first three modes of vibration of the clamped-simply supported beam as obtained using imaginary springs and the Lagrangian multiplier. The exact values are also given. As would be expected, as n is increased, the frequency parameters decrease both for any fixed value of K and for the Lagrangian multiplier solution. As the modulus of K is increased, for any particular value of n, the frequency parameters decrease for $-K$ and increase for $+K$. It can be seen that the imaginary spring results always bound the equivalent Lagrangian multiplier solution and the effect of approximating the rigid support by means of the imaginary springs is evident. In all cases, convergence is towards the exact solution. The $-K$ and Lagrangian multiplier solutions always yield upper bounds on the true values but that is not the case for $+K$ solutions. It may be noted that numerical problems began to be encountered when using nine terms in the series or when using values of the modulus of K of 10^9. However, convergence has been achieved to five figures for modulus of $K = 10^8$ and four figures for $n = 8$. Typically, a higher degree of convergence is achievable with increase in K than for increase in n before numerical problems are encountered. There will be exceptions to this, where the chosen shape functions extremely accurately approximate the true displacement of the system, but, in the experience of the writers, this is rarely the case for systems where the use of an approximate method of analysis is necessary.

2.2. *Natural frequencies of a stepped, simply supported beam*

In order to illustrate the behaviour of the solutions for a connected system, the stepped beam treated in reference [8] and shown in Figure 1(a) was considered. As in reference [8], it is modelled by two simply supported-free beams that are connected together by means of a translational and a rotational spring, each of very large stiffness, as illustrated in Figure 1(b). The only modification here is that the stiffness of the springs is permitted to be both positive and negative. In addition, an equivalent Lagrangian multiplier solution is given. For both solutions, the displacement functions are chosen as the simple polynomials

$$f_1(x_1) = \sum_{j=1}^{n} a_j (x_1/L_1)^j, \quad \text{for beam 1}, \tag{5a}$$

and

$$f_2(x_2) = \sum_{j=n+1}^{2n} a_j (x_2/L_2)^{j-n}, \quad \text{for beam 2}, \tag{5b}$$

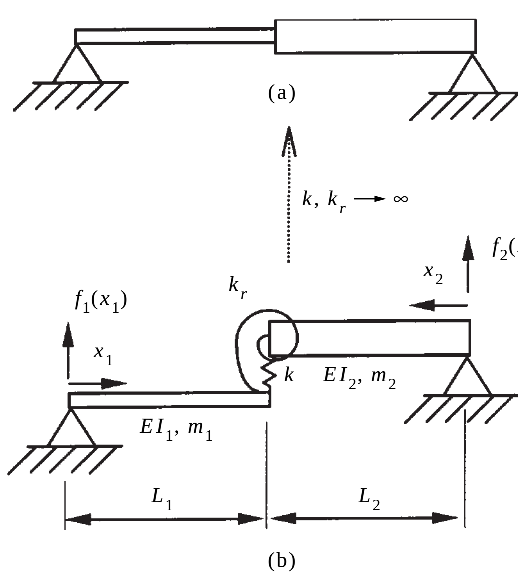

Figure 1. (a) A stepped simply supported beam and (b) its asymptotic model.

where n is the number of terms in the series and is taken to be the same for each section for convenience. The kinetic energy of the system is given by

$$\Psi = \int_0^{L_1} \frac{m_1}{2} (f_1(x_1))^2 \, dx_1 + \int_0^{L_2} \frac{m_2}{2} (f_2(x_2))^2 \, dx_2. \tag{6}$$

For the imaginary spring solution, the total potential energy of the system is given by

$$V = \int_0^{L_1} \frac{EI_1}{2} (f_1''(x_1))^2 \, dx_1 + \int_0^{L_2} \frac{EI_2}{2} (f_2''(x_2))^2 \, dx_2 + \tfrac{1}{2} k (f_1(L_1) - f_2(L_2))^2 + \tfrac{1}{2} k_r (f_1'(L_1) + f_2'(L_2))^2. \tag{7}$$

Substitution of equations (6) and (7) into the minimisation equation (2) again results in an eigenvalue matrix equation of the standard form.

For the Lagrangian multiplier solution, the deflection and slope continuity conditions at the joint require the constraint equations

$$f_1(L_1) - f_2(L_2) = 0, \qquad f_1'(L_1) + f_2'(L_2) = 0, \tag{8a, b}$$

be imposed and the quantities k and k_r in equation (7) be set to zero. Incorporating these conditions in the Rayleigh–Ritz minimisation equation gives

$$\partial V/\partial a_i - \omega^2 \partial \psi/\partial a_i + \partial (a_{n+1} \cdot (f_1(L_1) - f_2(L_2)))/\partial a_i + \partial (a_{n+2} \cdot (f_1'(L_1) + f_2'(L_2)))/\partial a_i = 0, \tag{9}$$

resulting in two additional rows and columns in the eigenvalue equation. The exact frequency equation for both the rigidly connected and spring connected beams was also derived from the differential equation.

Table 3 shows the first four natural frequency parameters $\lambda = (\rho A_1 L_1^4 \omega^2 / EI_1)^{1/4}$ for all the above cases for an assembly of beams of circular cross section having a diameter ratio (d_1/d_2) of 5. It can be seen that the frequency parameters for the models with positive stiffness obtained by using the Rayleigh–Ritz procedure are less than or equal to those for the corresponding rigid joint model obtained by using the Lagrangian multiplier method. For springs with negative stiffness the converse is true. The difference between the Rayleigh–Ritz results and the results for the Lagrangian multiplier method is due to the asymptotic approximation of the constraints using large stiffness parameter models. The difference between the results from the Lagrangian multiplier method and the exact solutions is due to the approximation of the actual model by a series of permissible functions as in the normal Rayleigh–Ritz procedure. In the absence of a Lagrangian multiplier solution, an upper limit of the error due to approximating rigid conditions may be estimated by taking the difference between the results for positive stiffness and the results for the corresponding negative stiffness.

The frequency parameters for the other diameter ratios for which results were reported in reference [8] were also computed using the positive and negative spring

TABLE 3

Natural frequency parameters for stepped beam, for $d_1/d_2 = 5$, $n = 5$ and $n = 7$

Mode	Method	Stiffness parameters 10^2	10^4	10^6	10^7	10^8	∞
1	Positive stiffness, $n = 5$	1·0600	1·0733	1·0734	1·0734	1·0734	
	Negative stiffness, $n = 5$	1·0876	1·0736	1·0734	1·0734	1·0734	
	Lagrangian multiplier, $n = 5$						1·0734
	Positive stiffness, $n = 7$	1·0660	1·0733	1·0734	1·0734	1·0734	
	Negative stiffness, $n = 7$	1·0876	1·0736	1·0734	1·0734	1·0734	
	Lagrangian multiplier, $n = 7$						1·0734
	Exact (positive stiffness)	1·0600	1·0733	1·0734	1·0734	1·0734	1·0734
	Exact (negative stiffness)	1·0876	1·0736	1·0734	1·0734	1·0734	1·0734
2	Positive stiffness, $n = 5$	3·4063	3·9593	3·9654	3·9654	3·9655	
	Negative stiffness, $n = 5$	4·4076	3·9716	3·9655	3·9655	3·9655	
	Lagrangian multiplier, $n = 5$						3·9655
	Positive stiffness, $n = 7$	3·4060	3·9573	3·9634	3·9634	3·9635	
	Negative stiffness, $n = 7$	4·4025	3·9695	3·9635	3·9635	3·9635	
	Lagrangian multiplier, $n = 7$						3·9635
	Exact (positive stiffness)	3·4060	3·9573	3·9634	3·9634	3·9634	3·9634
	Exact (negative stiffness)	4·4025	3·9695	3·9635	3·9635	3·9634	3·9634
3	Positive stiffness, $n = 5$	5·2230	7·0533	7·0934	7·0938	7·0938	
	Negative stiffness, $n = 5$	8·0738	7·1334	7·0942	7·0939	7·0938	
	Lagrangian multiplier, $n = 5$						7·0938
	Positive stiffness, $n = 7$	5·2152	7·0101	7·0454	7·0457	7·0457	
	Negative stiffness, $n = 7$	7·8022	7·0803	7·0461	7·0458	7·0458	
	Lagrangian multiplier, $n = 7$						7·0458
	Exact (positive stiffness)	5·2152	7·0093	7·0445	7·0449	7·0449	7·0449
	Exact (negative stiffness)	7·7960	7·0793	7·0452	7·0449	7·0449	7·0449
4	Positive stiffness, $n = 5$	8·2688	8·7906	8·7910	8·7910	8·7910	
	Negative stiffness, $n = 5$	8·8009	8·7910	8·7910	8·7910	8·7910	
	Lagrangian multiplier, $n = 5$						8·7910
	Positive stiffness, $n = 7$	7·9412	8·7398	8·7429	8·7430	8·7430	
	Negative stiffness, $n = 7$	8·7823	8·7457	8·7430	8·7430	8·7430	
	Lagrangian multiplier, $n = 7$						8·7430
	Exact (positive stiffness)	7·9338	8·7372	8·7405	8·7406	8·7406	8·7406
	Exact (negative stiffness)	8·7818	8·7435	8·7406	8·7406	8·7406	8·7406

models and the Lagrangian multiplier approach. Very close agreement was achieved between these results and those of reference [8] (they were identical for the spring of positive stiffness, since the solutions are identical) and the pattern was consistent with that found for the diameter ratio of 5.

3. CONCLUDING REMARKS

It has been illustrated that the Rayleigh–Ritz method with the asymptotic approach of modelling rigid boundary conditions or connections by means of "artificial" or "imaginary" springs can be improved by permitting the stiffness of these springs to be both positive and negative. By so doing, the bounds on the error introduced by the spring approximation may be determined, hence reducing the need to conduct extensive convergence studies for increase in spring stiffness. In addition, the useful effective upper bound characteristic of the Rayleigh–Ritz solution is maintained when negative spring stiffness is used.

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